Math 4 Honors
Lesson 8-4: *The Fundamental Theorem of Calculus*

Name ______________________________
Date ____________________________

Learning Goals:

- *I can use the General Power Rule for Integration to find the antiderivative of a function.*
- *I can use the Fundamental Theorem of Calculus to evaluate a definite integral.*
- *I can find the area between two curves using integration.*

Differential calculus was primarily concerned with the slope of a line tangent to a curve at a given point. This was helpful in a variety of problems including computing instantaneous velocity and acceleration. *Integral calculus is concerned with the area between that curve and the x-axis*. When you *differentiate* an equation you get the slope. When you *integrate* you get the area between equation and the x-axis. Up until this point, we have been approximating the area under irregular curves using Riemann Sums. Next, we will look at an algebraic method to help find the *exact values*.

The Fundamental Theorem of Calculus:

Must use "F" not "f"

Suppose that $f(x)$ is continuous on the interval $a \le x \le b$, and let $F(x)$ be an *antiderivative* of $f(x)$.

Then, $\boxed{\int_a^b f(x)dx = F(b) - F(a).}$

***Note: $F(b) - F(a)$ is called the net change from $x = a$ to $x = b$. It is abbreviated by the symbol: $F(x)\Big|_a^b$

Antiderivatives:

How do we find the antiderivative of a function? Study the next two examples of indefinite integrals & see if you can determine the process.

Example 1: Find antiderivative for the function, f(x) = 3x^2 + 7x.

Solution:

Step 1: Given function

$$f(x) = 3x^2 + 7x$$

$$\int f(x)\,dx = \int 3x^2 + 7x\,dx$$

Step 2: Separate the integral function

$$\int (3x^2 + 7x)\,dx = \int 3x^2\,dx + \int 7x\,dx$$

Step 3: Integrate each function with respect to ' x',

$$\int (3x^2 + 7x)\,dx = \frac{3x^3}{3} + \frac{7x^2}{2} + C$$

$$= x^3 + \frac{7x^2}{2} + C$$

Example 2: Find antiderivative for the function, f(x) = $\frac{2}{x^7}$

Solution:

Step 1: Given function

$$f(x) = \frac{2}{x^7}$$

$$\int f(x)\,dx = \int \frac{2}{x^7}\,dx$$

Step 2: Integrate the function $\frac{2}{x^7}$ with respect to ' x ',

$$\int \frac{2}{x^7}\,dx = \int 2x^{-7}\,dx$$

$$= 2\left(\frac{x^{-7+1}}{-7+1}\right)$$

$$= 2\left(\frac{x^{-6}}{-6}\right) + C$$

$$= -\frac{1}{3x^6} + C$$

How would you generalize the process?

What's up with "*C*"?

How can you determine if you have found the correct antiderivative?

OVER →

Differentiation and integration are inverses of each other. Page 2

$$\begin{tikzcd} f(x) \arrow[r, bend left, "Differentiate"] & f'(x) \arrow[l, bend left, "Integrate"] \end{tikzcd}$$

In case you weren't able to generalize the process . . . .

The General Power Rule for Integration: $F(x) = \int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

Examples: Integrate the functions with respect to x. In other words, find $F(x)$.
****Don't forget the "C"!*

1. $f(x) = 4x^2 + 2x - 3$

$F(x) = 4\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) - 3x + C$

$= \frac{4x^3}{3} + x^2 - 3x + C$

2. $f(x) = x^4 + 3x - 9$

$F(x) = \frac{x^5}{5} + \frac{3}{2}x^2 - 9x + C$

3. $\int \left[x^8 + x^{-8}\right] dx$

$F(x) = \frac{x^9}{9} + \frac{x^{-7}}{-7} + C$

$= \frac{x^9}{9} - \frac{1}{7x^7} + C$

4. $\int (5x^3 - 10x^{-6} + 4)\, dx$

$F(x) = 5\left(\frac{x^4}{4}\right) - 10\left(\frac{x^{-5}}{-5}\right) + 4x + C$

$= \frac{5x^4}{4} + \frac{2}{x^5} + 4x + C$

5. $\int \left[3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\right] dx =$

$\int \left(3x^{3/4} + 7x^{-5} + \frac{1}{6}x^{-1/2}\right) dx$

$F(x) = 3\left(\frac{x^{7/4}}{7/4}\right) + 7\left(\frac{x^{-4}}{-4}\right) + \frac{1}{6}\left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C$

$= \frac{12}{7}x^{7/4} - \frac{7}{4x^4} + \frac{1}{3}\sqrt{x} + C$

6. $\int dx$

$F(x) = x + C$

Some other special rules for integration:

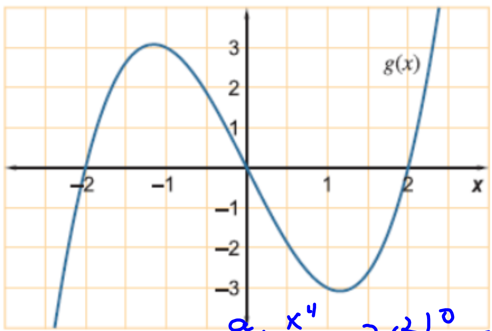
$$\int \sin x\,dx = -\cos x + c \qquad \int \cos x\,dx = \sin x + c$$

$$\int e^x\,dx = e^x + c \qquad \int a^x\,dx = \frac{a^x}{\ln a} + c \qquad \int \frac{1}{x}dx = \int x^{-1}\,dx = \ln|x| + c$$

Examples: Evaluate the following definite integrals.

1. Revisit problem #5 from Lesson 8-3 & use the **FTC** to answer parts a, b, & c. Compare these results to your original estimates.

The next diagram shows a graph of the function $g(x) = x^3 - 4x$.



$G(x) = \frac{x^4}{4} - 4\left(\frac{x^2}{2}\right) + c$

a. $\int_{-2}^{0} g(x)\,dx = \frac{x^4}{4} - 2x^2 + c$

b. $\int_{0}^{2} g(x)\,dx$

c. $\int_{-2}^{2} g(x)\,dx$

a. $\frac{x^4}{4} - 2x^2 \Big|_{-2}^{0} = G(0) - G(-2) = 0 - \left[\frac{16}{4} - 8\right] = 4$

b. $\frac{x^4}{4} - 2x^2 \Big|_{0}^{2} = G(2) - G(0) = \left[\frac{16}{4} - 8\right] - 0 = -4$

c. $\frac{x^4}{4} - 2x^2 \Big|_{-2}^{2} = G(2) - G(-2) = \left[\frac{16}{4} - 8\right] - \left[\frac{16}{4} - 8\right] = 0$

2. $\int_0^2 x^2 + 1\,dx$

$F(x) = \frac{x^3}{3} + x + c$

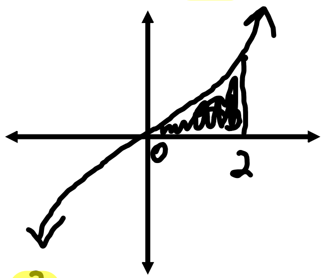
$\frac{x^3}{3} + x \Big|_0^2 = \frac{8}{3} + 2 - 0 = \boxed{\frac{14}{3}}$

3. $\int_0^{\pi} \sin(x)\,dx$

$F(x) = -\cos x + C$

$-\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos(0)) = -(-1) + 1 = \boxed{2}$

4. Find the area under the curve $y = x^3$ from $x = 0$ to $x = 2$.

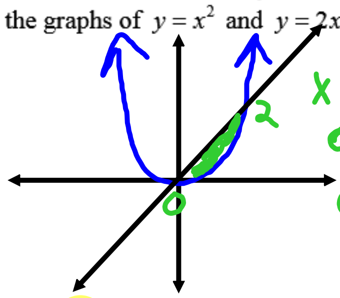


$\int_0^2 x^3\,dx$

$F(x) = \frac{x^4}{4} + C$

$\frac{x^4}{4}\Big|_0^2 = \frac{2^4}{4} - 0 = \boxed{4}$

5. Find the area bounded by the graphs of $y = x^2$ and $y = 2x$.



$x^2 = 2x$

$0 = 2x - x^2$

$0 = x(2 - x)$

$0 \qquad 2$

OVER →

$\int_0^2 (2x - x^2)\,dx$

$F(x) = \frac{2}{2}x^2 - \frac{x^3}{3} + C$

$x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \boxed{1.\overline{3}}$

HOMEWORK. *Please show all work on another piece of paper.*

Evaluate the following.

1. $\int_{-1}^{5}\left(\frac{2}{3}x+1\right)dx$
2. $\int_{-4}^{7}5dx$
3. $\int_{0}^{9}2xdx$
4. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x)dx$
5. $\int_{-3}^{8}\left(3x^2+4x+1\right)dx$
6. $\int_{0}^{5}e^x dx$
7. $\int_{4}^{8}8^x dx$
8. $\int_{0}^{7}\left(x^2+3x-2\right)dx$
9. $\int_{1}^{2}\left(\frac{4}{x^3}+\sqrt[3]{x^2}\right)dx$

Draw sketches for each of the following. Then solve.

10. Find the area under the curve $y=x^2+1$ from $x = 1$ to $x = 3$.

11. Find the area bounded by $y=x^2-1$ and the x-axis.

12. Find the area bounded by the graph of $y=x^3$, the x-axis, and the lines $x = -1$ and $x = 2$.

13. Find the area of the region bounded by $y=x^3-4x$ and $y = 0$.

14. Find the area of the region bounded by $y=x^2-4$ and $y=4-x^2$.

15. Find the area of the region bounded by $y=x^3-1$, $y = 0$, $x = 0$, and $x = 2$.

16. Find the area of the region bounded by $y=x^2$ and $y = 4$.

17. Find the area under the curve $y=\sqrt{x}$ from $x = 0$ to $x = 4$.

18. Find the area of the region bounded by $y=\sqrt{x}$, $x = 0$, and $y = 2$.

ANSWERS

1. 14
2. 55
3. 81
4. 2
5. 660
6. $\approx$147.41
7. $\approx$8,066,165.681
8. $\approx$173.83
9. $\approx$2.805
10. 32/3
11. 4/3
12. 17/4
13. 8
14. 64/3
15. 7/2
16. 32/3
17. 16/3
18. 8/3

Lesson 8-4 Homework

1. $F(x) = \frac{2}{3} \cdot \frac{x^2}{2} + x$

$= \frac{x^2}{3} + x + c$

$\frac{x^2}{3} + x \Big|_{-1}^{5} = \frac{5^2}{3} + 5 - \left[\frac{(-1)^2}{3} - 1\right]$

$= \frac{25}{3} + 5 - \frac{1}{3} + 1$

$= \boxed{14}$

2. $F(x) = 5x + c$

$5x + c \Big|_{-4}^{7} = 5(7) - (5)(-4)$

$= 35 + 20$

$\boxed{= 55}$

3. $F(x) = 2 \cdot \frac{x^2}{2} = x^2 + c$

$x^2 \Big|_0^9 = \boxed{81}$

4. $F(x) = \sin x + C$

$\sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)$

$= 1 - -1 = \boxed{2}$

5. $F(x) = 3 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} + x + c$

$x^3 + 2x^2 + x \Big|_{-3}^{8} = \left[8^3 + 2 \cdot 8^2 + 8\right] - \left[(-3)^3 + 2(-3)^2 - 3\right]$

$\boxed{= 660}$

6. $F(x) = e^x + c$

$e^x \Big|_0^5 = e^5 - e^0$

$\boxed{\approx 147.41}$

7. $F(x) = \frac{8^x}{\ln 8} + c$

$\frac{8^x}{\ln 8} \Big|_4^8 = \frac{8^8}{\ln 8} - \frac{8^4}{\ln 8}$

$\boxed{\approx 8066165.681}$

8. $F(x) = \frac{x^3}{3} + 3 \cdot \frac{x^2}{2} - 2x + c$

$= \frac{1x^3}{3} + \frac{3x^2}{2} - 2x + c$

$\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x \Big|_0^7 = \frac{1}{3} \cdot 7^3 + \frac{3}{2} \cdot 7^2 - 2 \cdot 7$

$\boxed{\approx 173.83}$

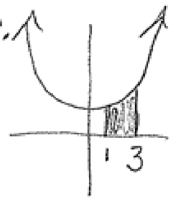
9. $f(x) = 4x^{-3} + x^{\frac{2}{3}}$

$F(x) = 4 \cdot \frac{x^{-2}}{-2} + \frac{x^{\frac{5}{3}}}{\frac{5}{3}}$

$= -2x^{-2} + \frac{3x^{5/3}}{5} + c$

$-2x^{-2} + \frac{3}{5}x^{\frac{5}{3}} \Big|_1^2 = -2(2)^{-2} + \frac{3}{5} \cdot 2^{\frac{5}{3}} - \left[-2 + \frac{3}{5}\right]$

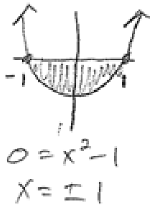
$\boxed{\approx 2.805}$

10. 

$\int_1^3 (x^2+1)dx =$

$\frac{1}{3}x^3+x \Big|_1^3 =$

$(\frac{27}{3}+3)-(\frac{1}{3}+1) = \boxed{\frac{32}{3}}$

11. 

$-\int_{-1}^1 (x^2-1)dx$

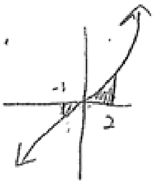
$0=x^2-1$

$x=\pm 1$

$\int_{-1}^1 (1-x^2)dx = x-\frac{1}{3}x^3\Big|_{-1}^1$

$=(1-\frac{1}{3})-(-1+\frac{1}{3})$

$\boxed{=\frac{4}{3}}$

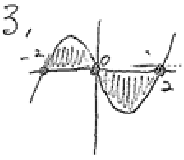
12. 

$-\int_{-1}^0 x^3dx+\int_0^2 x^3dx=$

$\int_{-1}^0 -x^3 + \int_0^2 x^3dx =$

$-\frac{1}{4}x^4\Big|_{-1}^0+\frac{1}{4}x^4\Big|_0^2=$

$(0+\frac{1}{4})+(\frac{16}{4}+0)=\boxed{\frac{17}{4}}$

13. 

$\int_{-2}^0(x^3-4x)dx+\int_0^2(4x-x^3)dx$

$\frac{1}{4}x^4-2x^2\Big|_{-2}^0+2x^2-\frac{1}{4}x^4\Big|_0^2$

$x^3-4x=0$

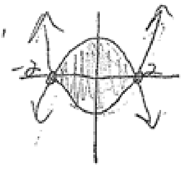
$x(x^2-4)=0$

$x=0$ $\quad x=\pm 2$

$0-(\frac{16}{4}-8)+(2\cdot4-\frac{1}{4}\cdot16-0)=$

$0+4+$

$\boxed{8}$

14. 

$\int_{-2}^2[4-x^2-(x^2-4)]dx=$

$\int_{-2}^2(8-2x^2)dx=$

$8x-\frac{2}{3}x^3\Big|_{-2}^2=$

$16-\frac{2}{3}\cdot8-(-16-\frac{2}{3}\cdot-8)=$

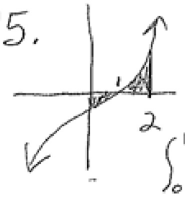
$\boxed{\frac{64}{3}}$

$x^2-4=4-x^2$

$2x^2-8=0$

$2(x^2-4)=0$

$x=\pm 2$

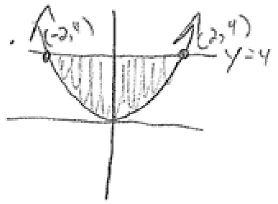
15. 

$-\int_0^1(x^3-1)dx+\int_1^2(x^3-1)dx=$

$\int_0^1(1-x^3)dx+\int_1^2(x^3-1)dx=$

$x-\frac{1}{4}x^4\Big|_0^1+\frac{1}{4}x^4-x\Big|_1^2=$

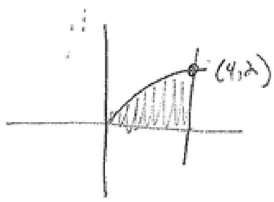
$1-\frac{1}{4}-0+\frac{1}{4}\cdot16-2-(\frac{1}{4}-1)=\boxed{\frac{7}{2}}$

16. 

$\int_{-2}^2(4-x^2)dx=$

$4x-\frac{1}{3}x^3\Big|_{-2}^2=$

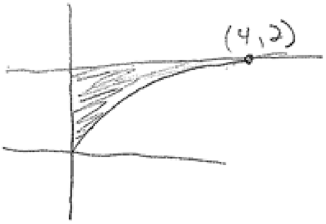
$8-\frac{8}{3}-(-8+\frac{8}{3})=\boxed{\frac{32}{3}}$

17. 

$\int_0^4(\sqrt{x})dx=$

$\frac{2}{3}x^{3/2}\Big|_0^4=$

$\frac{2}{3}\cdot4^{3/2}-0=\boxed{\frac{16}{3}}$

18. 

$\int_0^4(2-\sqrt{x})dx=$

$2x-\frac{2}{3}x^{3/2}\Big|_0^4=$

$8-\frac{2}{3}\cdot4^{3/2}-0=\boxed{\frac{8}{3}}$